Technical Notes

An Axially Compressed, Cylindrical Shell with a Viscoelastic Core

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Nomenclature

a, h, L = shell radius, thickness, and length, respectively
E, γ = Young's modulus and Poisson's ratio, respectively
G_0, G_1 = constants in the shear modulus operator for the viscoelastic core
I_0, I_1 = Bessel functions
p = outward radial pressure on the shell
P = $P(t)$ = axial compression of the shell
P_0 = magnitude of the step-applied axial compression
t = time
u, w = axial and radial shell displacements, respectively
x, ϕ = axial and polar coordinates, respectively
γ, τ = shear strain and stress, respectively
σ_x = axial stress in the shell
$(\ )'$ = $a\partial(\ )/\partial x$
D = $E_sh/(1 - \nu^{-2})$
k = $h^2/12a^2$

Subscripts

c = core
s = shell
M = membrane problem
B = buckling problem
m = index of the Fourier series

UNDER conditions of axial compression, thin monocoque cylindrical shells that are designed primarily as pressure vessels are subject to failure by buckling. Although the tensile strength allowables have increased with the use of new materials in modern missiles, the buckling strength has not increased accordingly. Consequently, stability considerations have assumed increasing importance in the design of modern missiles and space vehicles. With the use of solid propellants, it is desirable to establish the extent to which the propellant serves to stabilize the structure and thus increase the critical buckling load. A number of papers appearing recently in the literature (e.g., Refs. 1 and 2) have treated the buckling of thin cylindrical shells filled with an elastic core. To incorporate the time-dependent behavior of the propellant, an analysis of the stability under axial compression of finite cylinders enclosing a viscoelastic core is considered. Expressions for the radial displacements and axial stresses are obtained, the ends of the shell being assumed to be fixed in the radial direction.

The analysis is limited to small quasi-static and axisymmetric deformations of a thin, cylindrical shell enclosing a solid viscoelastic core and loaded symmetrically in the axial direction. The ends of the shell are assumed to be fixed in the radial direction. The core material is assumed to be incompressible ($\nu_c = \frac{1}{2}$) and to be viscoelastic in shear distortion:

$$\tau_c = (G_0 + G_1\,\partial/\partial t)\gamma_c \tag{1}$$

The surface tractions between shell and core are neglected and the radial pressure between shell and core is assumed to be a linear function $p(w)$ of the radial displacement w (see Seide[1]).

The analysis of the problem is divided into two parts: the buckling deformation problem and the membrane shell problem. The equation for the membrane problem is the following[3]:

$$(1 - \nu_s^2)w_M/a = [ap(w_M) + \nu_sP]/D \tag{2}$$

The following equations for the buckling problem are taken from Flügge[3]:

$$Au_B'' + Bw_B' - kw_B''' = 0 \tag{3a}$$

$$Bu_B' - ku_B''' + Cw_B + (P/D)w_B'' + kw_B'''' - ap(w_B)/D = 0 \tag{3b}$$

$$A = 1 - P/D \approx 1 \tag{3c}$$

$$B = \nu_s + ap(w_M)/D \approx \nu_s \tag{3d}$$

$$C = 1 + k \approx 1 \tag{3e}$$

The boundary conditions require that there be no moments or radial deflections at the edge of the shell. Also, it is required that the axial compression remain $P(t)$ in spite of any buckling deformations. The initial condition requires that the shell be undeformed at $t = 0$:

$$w = w'' = 0 \tag{4a}$$

$$uB'A + \nu_swB + kwB'' = 0 \qquad \text{at } x = 0, L \tag{4b}$$

$$w = 0 \text{ at } t = 0 \tag{4c}$$

The quantities P/D, $ap(w_B)/D$, and k are quite small; so, the approximate expressions for A, B, and C given in Eqs. (3c–3e) will be used. After this simplification, (3a, 3b, and 4b) can be combined into one equation:

$$w_B\,(1 - \nu_s^2) + (2\nu_sk + P/D)w_B'' + kw_B''' - ap(w_B)/D = 0 \tag{5}$$

The buckling deformation problem and the membrane shell problem can now be combined. Equations (2) and (5) yield an expression for the total radial displacement of the shell:

$$w = w_B + w_M$$

$$w(1 - \nu_s^2) + (2\nu_sk + P/D)w'' + kw'''' - ap(w)/D = w_M(1 - \nu_s^2) - ap(w_M)/D = \nu_s\,P/D \tag{6}$$

Unlike w or w_B, the displacement w_M is a slowly varying function of x, and, therefore, w_M'''' and w_M'' will be about the same magnitude as w_M. This means that kw_M'''' and $(2\nu_sk + P/D)w_M''$ are small compared to $(1 - \nu_s^2)w_M$; consequently, they were neglected in (6).

Seide[1] developed an expression for the radial pressure between a cylindrical shell and an elastic core. His expression was extended to the viscoelastic core of this note by setting

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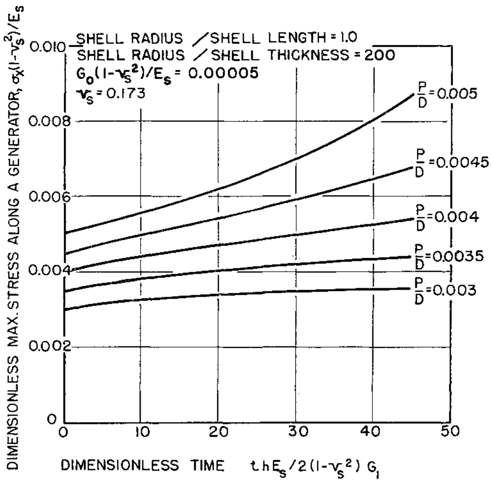

Fig. 1 Maximum stress along any shell generator vs time.

$\nu_c = \frac{1}{2}$ and replacing the shear modulus of the elastic core, $E_c/2(1 + \nu_c)$, with the shear modulus operator of (1):

$$p(w) = -\sum_{m=1}^{\infty} 2f(\lambda)\left[\left(\frac{G_0 w_m}{a}\right) + \frac{G_1 \partial(w_m/a)}{\partial t}\right]\sin\left(\frac{\lambda x}{a}\right) \quad (7a)$$

$$\lambda = m\pi a/L \quad (7b)$$

$$w = \sum_{m-1}^{\infty} w_m \sin\left(\frac{\lambda x}{a}\right) \quad (7c)$$

$$f(\lambda) = \left[\frac{\lambda I_0(\lambda)}{I_1(\lambda)}\right]^2 - 1 - \lambda^2 \quad (7d)$$

This expression for $p(w)$ and its Fourier series expansion of w is substituted into (6), giving a linear, first-order, ordinary, differential equation for w_m. It would be a simple matter to express w_m as an integral if desired:

$$(w_m/a)[\lambda^4 k - \lambda^2(2\nu_s k + P/D) + 1 - \nu_s^2 + 2f(\lambda)aG_0/D] + (2aG_1/D)d(w_m/a)dt = \begin{cases} 0 \text{ if } m \text{ is even} \\ 4a\nu_s P/D\lambda L \text{ if } m \text{ is odd} \end{cases} \quad (8)$$

The boundary condition (4b) is satisfied by (6). The Fourier series expansion of w satisfies (4a). The initial condition (4c) will be satisfied if

$$w_m = 0 \text{ at } t = 0 \quad (9)$$

Expressions (7d, 8, and 9) give the deformations, and these, in turn, can be used to calculate the stress σ_x at the outside or inside surface of the shell:

$$\alpha_x = \frac{-P}{D} \pm \frac{(3k)^{1/2}w''}{a} = \frac{-P}{D} \pm (3k)^{1/2}\sum_{m=1}^{\infty}\left(\frac{\lambda^2 w_m}{a}\right)\sin\left(\frac{\lambda x}{a}\right) \quad (10)$$

This stress was computed for an example problem, the parameters given below being used. For simplicity, only the largest stress σ_x was computed:

$$\mathrm{P}(t) = \begin{cases} 0 \text{ for } t < 0 \\ P_0 \text{ for } t \geq 0 \end{cases}$$

$$G_0(1 - \nu_s^2)/E_s = 0.00005 \qquad \nu_s = 0.1732$$

$$a/L = 0.5 \qquad a/h = 200$$

For a load P_0 below a certain critical value, the stress α_x approaches a limiting value as time increased (i.e., as $t \to +\infty$). For loads above this critical value, the stresses increase exponentially with time. The increase is more rapid with larger loads (see Fig. 1).

The critical load is the "long-time" buckling load of the case. If loads were held constant for a long period of time, one could neglect the viscoelastic characteristics of the core and treat it as elastic body. The buckling or Euler load of a case with this elastic core would be the "critical load" just mentioned.

The solid viscoelastic core used in this calculation stiffened the case considerably, and the results are not generally applicable to a practical situation. It is recognized that the more realistic case would have been for a hollow cylindrical core; however, the method of incorporating the viscoelastic model has been demonstrated as was the intent of this note. A core with a concentric hole can be incorporated by defining a new $f(\lambda)$. The arduous details of this change can be found in Seide's paper.[1]

References

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[2] Yao, J. C., "Buckling of axially compressed long cylindrical shell with elastic core," J. Appl. Mech. **29**, 329–334 (1962).

[3] Flugge, W., *Stresses in Shells* (Springer-Verlag, Berlin, 1960), Chap. 3, p. 130 and Chap. 7, p. 422.

One-Dimensional Flow Considering Buoyancy Forces

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Nomenclature

T = absolute temperature
C_p = specific heat at constant temperature
C_v = specific heat at constant volume
γ = ratio of the specific heats
P_γ = Prandtl number
g = acceleration due to gravity

Subscripts

0 = condition at $x = 0$
α = conditions at $x \to \infty$
μ = coefficient of viscosity of the fluid
β = coefficient of volume expansion
K = thermal conductivity of the fluid

I. Introduction

An exact solution of nonlinear equations for viscous and heat conducting compressible fluid has been given by Morduchow and Libby[1], Pai[2], and Ludford[3] for one-dimensional flow. They neglected buoyancy forces; but in motions where temperature differences change density, it becomes necessary to include buoyancy forces in the equation of motion of a viscous fluid, and they should be treated as impressed body forces. The buoyancy forces are caused by changes in volume which are associated with the temperature differences. Now, if β is the coefficient of volume expansion, and if $\theta = T - T_\infty$ is the temperature difference between a hotter fluid particle and the colder surroundings, the change in volume per unit

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